Section 6. Boundedness of Complements

Def Let $(X', B'+M')$ be a proj generalized pair with data

$\phi: X \to X'$ and M a nef divisor on X, $\phi_* M = M'$.

An n-complement of $K_{X'} + B' + M'$ is $K_{X'} + B'^{+} + M'$ for some divisor B'^{+} such that (1) $(X', B'^{+} + M')$ is glc

(2) $n(K_{X'} + B'^{+} + M') \sim 0$ and nM is b-Cartier

(3) $nB'^{+} \geqslant n\lfloor B' \rfloor + \lfloor (n+1)\{B'\} \rfloor$

(For monotone complements, (3') $B'^{+} \geqslant B'$.)

Goal: Boundedness of relative complements for pairs (Thm 1.8)

$+$ Boundedness of exceptional pairs (Thm 1.11)

$\Rightarrow$ Boundedness of complements for generalized pairs (Thm 1.10)

Thm 1.8 (Boundedness of relative complements for pairs)

$d \in \mathbb{Z}_{>0}$, $\mathcal{R} \subset [0,1] \cap \mathbb{Q}$ finite. Then $\exists\, n = n(d, \mathcal{R})$ satisfying:

$\ulcorner$ Assume (X, B) a pair and $X \to Z$ a contraction such that

- (X, B) is lc of dim d, $\dim Z > 0$
- $B \in \Phi(\mathcal{R}) = \{1 - \frac{r}{m} : r \in \mathcal{R},\ m \in \mathbb{Z}_{>0}\}$
- X is Fano type over Z
- $-(K_X + B)$ is nef$/Z$.

Then for any closed point $z \in Z$, $\exists$ n-complement $K_X + B^{+}$ of $K_X + B$ over z. $\lrcorner$

Thm 1.10 (Boundedness of complements for generalized pairs)

$d, p \in \mathbb{Z}_{>0}$. $\mathcal{R} \subset [0,1] \cap \mathbb{Q}$ finite. Then $\exists\, n = n(d, p, \mathcal{R})$ satisfying:

$\ulcorner$ Assume $(X', B'+M')$ is a proj g-pair with data $\phi: X \to X'$ and M, s.t.

- $(X', B'+M')$ is glc of dim d
- $B' \in \Phi(\mathcal{R})$, pM is b-Cartier
- X' is of Fano type
- $-(K_{X'} + B' + M')$ is nef.

Then $\exists$ n-complement B'^{+} of B'. $\lrcorner$

Thm 1.11 (Boundedness of exceptional pairs)

$d, p \in \mathbb{Z}_{>0}$, $\mathcal{R} \subset [0,1] \cap \mathbb{Q}$ finite. Consider all g-pairs in Thm 1.10 which are exceptional (i.e., $\forall \Delta' \geq 0$ s.t. $K_{X'} + B' + \Delta' + M' \sim_{\mathbb{Q}} 0$, $(X', B' + \Delta' + M')$ is gklt)

Then the set of (X', B') is log bounded.

Idea of 1.8 + 1.11 $\Rightarrow$ 1.10

- X' is Fano type, hence a MDS. nef $\Rightarrow$ semiample.
- Let $X' \xrightarrow{f} Z'$ be contraction defined by $-(K_{X'}+B'+M')$

Case I $\dim Z'>0$, M' not big $/ Z'$

Prove a canonical bundle formula $K_{X'}+B'+M' \sim f^*(K_{Z'}+B_{Z'}+M_{Z'})$

$\dim Z' < \dim X'$, $\Rightarrow$ by induction on $\dim X'$, can find a n-complement $B_{Z'}^+$ of $B_{Z'}$.

Pullback $B_{Z'}^+$ to get a complement B'^+ of B'.

(Prop 6.3 + Prop 6.5).

Case II $\dim Z'=0$, M' not big $/ Z'$

Run MMP to assume that $M' \sim_{\mathbb{Q}} 0$ over Z'.

$K_{X'}+B'+M' \sim_{\mathbb{Q}} 0 \Rightarrow$ we only need to bound Cartier index.

(Prop. 6.10).

Case III M' big $/ Z'$.

Run MMP to assume M' is nef $/ Z'$.

Use positivity of M' to perturb $(X', B'+M')$ to a g-pair $(X', T'+\alpha M')$ which is a gplt pair and log Fano with $\alpha < 1$.

$S = \lfloor T' \rfloor$ irred and $S \subseteq \lfloor B' \rfloor$.

Apply divisorial adjunction to S and lift complements from S using KV vanishing. (Prop 6.7).

Properties of complements

Let $(X', B'+M')$ proj g-pair with $\phi: X \to X'$ and M.

(1) Assume $\exists\, B'^{+} \geq B'$ s.t. $(X', B'^{+}+M')$ is glc, nM is b-Cartier,

$n(K_{X'}+B'^{+}+M') \sim 0$.

Then $nB'^{+} \geq n\lfloor B' \rfloor + \lfloor (n+1)\{B'\} \rfloor \Rightarrow B'^{+}$ is n-compl. of B.

(2) Let $X' \dashrightarrow X''$ be a $(K_{X'}+B'+M')$ - nonnegative contraction.

If $K_{X''}+B''+M''$ has n-complements, then so does $K_{X'}+B'+M'$.

Pf

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ X' \arrow[rr, dashed] & & X'' \end{tikzcd}$$

Write $\phi^*(K_{X'}+B'+M')+P = \psi^*(K_{X''}+B''+M'')$

$P \geq 0$.

Let B''^{+} be a n-compl. of B''. Then define

$B'^{+} := B' + \phi_*(P+\psi^*(B''^{+}-B''))$ so that

$\phi^*(K_{X'}+B'^{+}+M') \sim \psi^*(K_{X''}+B''+M'') \sim_{\mathbb{Q}} 0$

$\Rightarrow B'^{+}$ is n-compl. of B'. $\square$

Prop 6.3 (Canonical Bundle Formula)

$d \in \mathbb{Z}_{>0}$, $\mathfrak{R} \subset [0,1] \cap \mathbb{Q}$ finite. Assume Thm 1.8 in dim d.

Then $\exists\, q \in \mathbb{Z}_{>0}$ and a finite set $\mathcal{S} \subseteq [0,1] \cap \mathbb{Q}$ depending only on d, $\mathfrak{R}$, satisfying

⌈ Assume (X, B) a pair, $f: X \to Z$ a contraction s.t.

- (X, B) proj. lc of dim d, $\dim Z > 0$
- $K_X + B \sim_{\mathbb{Q}} 0$ over Z
- $B \in \Phi(\mathfrak{R})$
- X is Fano type over some non-empty open $\mathcal{U} \subseteq Z$
- The generic point of each lc center of (X, B) maps into $\mathcal{U}$.

Then we can write $q(K_X + B) \sim q f^*(K_Z + B_Z + M_Z)$ s.t.

- B_Z, M_Z are disc and moduli part of adjunction
- $B_Z \in \Phi(\mathcal{S})$
- $q M_{Z'}$ is nef Cartier on some bir. model $Z' \to Z$. ⌋

Rmk. B_Z is uniquely defined

M_Z is defined up to $\mathbb{R}$ linear equivalence.

Step 1-4 of proof: prove a weaker version where only require

$q M_Z$ is integral (no nefness condition).

Pf. Step 1: Find q and construct M_Z.

- Thm 1.8: $\exists\, q = q(d, \mathcal{R})$ s.t. $K_X + B$ has q-complements $K_X + B^+$ over a fixed $z \in Z$.

- Fix $z \in U$.

Since $K_X + B \sim_{\mathbb{Q}} 0 \sim_{\mathbb{Q}} K_X + B^+$ over $z \in Z$.

$\Rightarrow$ $B^+ = B$ over $z \in Z$.

$\Rightarrow$ $B^+ = B$ over η_Z.

$\Rightarrow$ $q(K_X+B) = \overbrace{q(K_X+B^+)}^{\sim 0}$ over η_Z

$\Rightarrow$ $\exists\, \alpha \in K(X)$ s.t. $\underbrace{q(K_X+B) + \mathrm{Div}(\alpha)}_{\shortparallel \atop qL} = 0$ over η_Z.

Now $L \sim_{\mathbb{Q}} K_X + B \sim_{\mathbb{Q}} 0$ over Z, so $L = f^* L_Z$ for some L_Z on Z.

Define $M_Z := L_Z - (B_Z + M_Z)$, so that

$$q(K_X+B) \sim qL = q f^* L_Z = q f^*(K_Z + B_Z + M_Z).$$

Rmk. Can define $M_{Z'}$ for $Z' \to Z$ using pullbacks of L_Z.

Check $\{ M_{Z'} : Z' \to Z \}$ determines a b-divisor on Z.

Step 2 We reduce to the case $\dim Z = 1$. by cutting Z with hyperplane sections.

Assume $\dim Z > 1$. Let H be a general hyperplane section of Z.

Let $G := f^*H$.

Write $K_G + B_G := (K_X + B + G)|_G$ for some B_G.

Claim: The pair (K_G, B_G) satisfyies:

(1) (K_G, B_G) is lc of dim $d-1$.

(2) $K_G + B_G \sim_{\mathbb{Q}} 0$ over H.

(3) $B_G \in \Phi(\mathfrak{R}')$ for some finite $\mathfrak{R}'$. (Lemma 3.3)

(4) G is Fano type over $U \cap H$.

(5) generic pt of every lc center of (G, B_G) maps into $U \cap H$.

Write $K_G + B_G \sim g^*(K_H + B_H + M_H)$, where $g: G \to H$, then

(6) $\mathrm{coeff}(B_Z) \subseteq \mathrm{coeff}(B_H)$.

(7) We can choose M_H so that

$$\mathrm{coeff}(M_Z) \subseteq \mathrm{coeff}(M_H).$$

Pf (6) Let D be a prime div on Z, C a component of $D \cap H$.

Let $t = \mathrm{lct}(X, B)$ wrt f^*D over η_D.

$\Rightarrow$ $\exists$ lc center of $(X, B + tf^*D)$ mapping onto D.

$\Rightarrow$ " $(X, B + G + tf^*D)$ "

$\Rightarrow$ $\exists$ " $(G, B_G + tg^*C)$ mapping onto C.

$\Rightarrow$ $t = \mathrm{lct}(G, B_G)$ wrt g^*C over η_C

$\Rightarrow$ $\mathrm{coeff}_D(B_Z) = 1 - t = \mathrm{coeff}_C(B_H)$.

(7) Pick a general $H' \sim H$. Let $K_H := (K_Z + H')|_H$.

Define $M_H := (L_Z + H')|_H - (K_H + B_H)$, so that

$$\begin{cases} q(K_G + B_G) \sim q g^*(K_H + B_H + M_H) \\ B_H + M_H = (B_Z + M_Z)|_H. \end{cases}$$

$\Rightarrow \mathrm{coeff}_C(B_H + M_H) = \mathrm{coeff}_D(B_Z + M_Z)$.

$\Rightarrow \mathrm{coeff}_C(M_H) = \mathrm{coeff}_D(M_Z)$. $\square$

<u>Step 3</u> Show existence of S when $\dim Z = 1$.

Let $t = \mathrm{lct}(X, B)$ wrt f^*z over $z \in Z$.

<u>Claim</u>: $\exists$ q-compl. B^+ of B over z s.t. $B^+ = B + tf^*z$ over z.

Sketch: Let $\Gamma := B + tf^*z$. $\Rightarrow K_X + \Gamma \sim_{\mathbb{Q}} 0$ over Z.

If $\mathrm{coeff}(\Gamma) \subseteq \Phi(\mathcal{R})$, then Thm 1.8 $\Rightarrow \exists$ q-compl B^+ of $K_X + \Gamma$ over z.

$B^+ - B \sim_{\mathbb{Q}} 0$ over z $\Rightarrow B^+ = B + sf^*z$. for some s.

$$\left.\begin{array}{l} \cdot\ B^+ \geq \Gamma \Rightarrow s \geq t. \\ \cdot\ (X, \Gamma) \text{ has a lc center over } z \Rightarrow s \leq t. \end{array}\right\} \Rightarrow s = t.$$

Pick S a component of f^*z. Let b, b^+, m be coeff of S in B, B^+, f^*z.

- $m \in \mathbb{Z}_{>0}$, $b^+ = b + tm$, $b \in \Phi(\mathcal{R})$, $qb^+ \in \mathbb{Z}$.

$\Longrightarrow \mathrm{coeff}_z(B_Z) = 1 - t \in \Phi(\mathcal{S})$ for some finite $\mathcal{S}$.

Step 4 We prove qM_Z is integral when $\dim Z = 1$.

- Step 1: $B^+ = B$ over η_Z.

$\Rightarrow q(K_X+B) = q(K_X+B^+)$ over η_Z

$\Rightarrow q(K_X+B) \sim 0$ over some open $V \subseteq Z$.

Assume $\operatorname{supp} B_Z \subseteq Z \setminus V$.

- Define $\Theta := B + \sum_{z \in Z\setminus V} t_z f^*z$, $t_z = \operatorname{lct}(X, B)$ wrt f^*z over z.

Then $\Theta_Z :=$ disc part of $K_X + \Theta$ over Z.

$= B_Z + \sum_{z\in Z\setminus V} t_z \cdot z$. is integral

By Claim in Step 3, Θ is a q-compl of B over $\forall z \in Z \setminus V$.

$\underset{(\overset{\shortparallel}{B})}{\Theta}$ is a q-compl of B inside V

$\Rightarrow \Theta$ is a q-compl of B over Z.

- Now $\underbrace{q(K_X+\Theta)}_{\text{integral}} = q f^*(K_Z + \underset{\uparrow}{\Theta_Z} + \underset{\uparrow}{M_Z})$.

integral

$\Rightarrow qM_Z$ is integral.

<u>Step 5</u> $qM_{Z'}$ is nef Cartier on some $Z' \to Z$.

<u>Idea</u>: Nefness follows from Thm 3.6 for suff. high resol $Z' \to Z$.

Integrability: want to reduce to case in Step 1-4, i.e.

want to construct (X', Δ') s.t.

(1)
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Z' \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$
✓

(2) $\mathrm{coeff}(\Delta') \subseteq \Phi(\mathcal{R})$ ✓

(3) $K_{X'} + \Delta' \sim_{\mathbb{Q}} 0$ over Z' ✓

(4) $K_{X'} + \Delta' \sim_{\mathbb{Q}} f^*(K_{Z'} + \Delta'_{Z'} + M_{Z'})$. ✓

<u>Obs</u>: If (X', Δ') is a dlt mod. of (X, B) and satisfies (1),

then it satisfies (1) - (4).

Take X' satisfying (1), $\Delta' := \widetilde{B} + ex(X' \to X)$.

Run MMP on $K_{X'} + \Delta'$ over $X \times_Z Z'$

Replace X' by min model

Z' by can. model

$\square$

Prop 6.5 (Lifting complements from base of fibration)

Assume Thm 1.10 in dim $\leq d-1$ (for gpairs) and Thm 1.8 in dim d (relative pairs).

Then Thm 1.10 holds in dim d for $(X', B'+M')$ s.t. there is a contraction $X' \to V'$ satisfying

- $K_{X'} + B' + M' \sim_{\mathbb{Q}} 0$ over V'
- $\dim V' > 0$
- M' is not big $/V'$.

Sketch: Step 1: Reduce to case $M' \sim_{\mathbb{Q}} 0$ over V'.

Run MMP on M'/V' to assume that M' is semiample $/V'$.

Replace $X' \to V'$ by the contraction induced by M'. to assume $M' \sim_{\mathbb{Q}} 0/V'$

Step 2 $K_{X'} + B' \sim_{\mathbb{Q}} 0$. use Prop 6.3 to write

$$q(K_{X'} + B') \sim qf^*(K_{V'} + B_{V'} + P_{V'})$$

$B_{V'} \in \Phi(S)$ and $qP_{V'}$ is b-nef Cartier.

Step 3 Construct a diagram

$$\begin{tikzcd} M \arrow[d] & X \arrow[r] \arrow[d] & V \arrow[d] & M_V \arrow[d] \\ M' & X' \arrow[r] & V' & M_{V'} \end{tikzcd}$$

such that

(Lemma 2.44).

- pM_V is nef Cartier
- $pM \sim pf^*M_V$
- $qM' \sim qf'^*M_{V'}$

Step 4 We can write

$$q(K_{X'}+B'+M') \sim q f^*(K_{V'}+B_{V'}+P_{V'}+M_{V'})$$

- Prove that $(V', B_{V'}+P_{V'}+M_{V'})$ is glc.
- Apply 1.10 to get a q-compl. $B^+_{V'}$ of $B_{V'}$.
- Pullback $B^+_{V'}$ to get a q-compl. B'^+ of B'.

Prop 6.7 (Lifting complements from divisorial adjunction)

Assume Thm 1.10 in dim $d-1$. Then Thm 1.10 holds in dim d for $(X', B'+M')$ such that:

- $B' \in \mathcal{R}$ (∇ not $B' \in \Phi(\mathcal{R})$)
- $(X', \Gamma' + \alpha M')$ is $\mathbb{Q}$-factorial gplt for some $\Gamma' \geq 0$ and $\alpha \in (0,1)$
- $S' = \lfloor \Gamma' \rfloor$ is irred and $S' \subseteq \lfloor B' \rfloor$.
- $-(K_{X'} + \Gamma' + \alpha M')$ is ample.

Sketch proof (existence of monotone complements)

Step 1. Apply adjunction on S' and find n-compl. using Thm 1.10 in dim $d-1$.

Suppose $\phi: X \to X'$ is a high resol of $(X', B'+\Gamma')$ s.t.

$\psi: S \to S'$ is a morphism.

pM is Cartier $\Rightarrow$ can pick M_S s.t.

$$p(K_{S'} + B_{S'} + M_{S'}) \sim p(K_{X'} + B' + M')|_{S'} \sim_{\mathbb{Q}} 0.$$

$\xRightarrow{\text{Thm 1.10 } (d-1)}$ $B_{S'}^+$ of $B_{S'}$ an n-compl.

such that $nB_{S'}^+$ and $nM_{S'}$ are integral.

Step 2. Write $\underbrace{-n(K_X + B + M)}_{\mathcal{L}} := -n\phi^*(K_{X'} + B' + M')$

$$K_X + \Gamma + \alpha M := \phi^*(K_{X'} + \Gamma' + \alpha M').$$

Let $P := -\lfloor \Gamma^{<0} \rfloor \geq 0$ so that $(X, \Gamma + P)$ is plt and $\lfloor \Gamma + P \rfloor = S$.

Furthermore, P is exceptional $/ X'$.

<u>Step 3</u> Use KV vanishing to lift sections.

Note $$L+P = K_X + \underbrace{(\Gamma+P)}_{S+(\text{frac})} + \underbrace{\underbrace{(-K_X-\Gamma-\alpha M)}_{\phi^*(\text{ample})} + \underbrace{\alpha M + L}_{\text{nef}}}_{\text{big \& nef}}.$$

$$\overset{\text{KV vanishing}}{\Longrightarrow} \quad H^1(L+P-S) = 0.$$

$$\Longrightarrow \quad H^0(L+P) \twoheadrightarrow H^0((L+P)|_S).$$

This Gives a lift of a compl. $B_S^{+'}$ on the base to some div B'^{+} on X'.

<u>Step 4</u> Check that $(X', B'^{+} + M')$ is glc

and hence B'^{+} is a n-compl. of B. $\square$.